Spin Decoherence in a Gravitational Field

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Abstract

We discuss a mechanism of spin decoherence in gravitation within
the framework of general relativity. The spin state of a particle moving
in a gravitational field is shown to decohere due to the curvature of
spacetime. As an example, we analyze a particle going around a static
spherically-symmetric object.

1 Introduction

The spin of a particle is an interesting degree of freedom in quantum theory.
Recently, Peres, Scudo, and Terno [1] have shown that in special relativity
the spin entropy (i.e., the von Neumann entropy of a spin state) of a particle
is not invariant under Lorentz transformations unless the particle is in a
momentum eigenstate. Namely, even if the spin state is pure in one frame
of reference, it may become mixed in another frame of reference. The origin
of this spin decoherence is that the Lorentz transformation entangles the
spin and momentum via the Wigner rotation [2]. The entanglement then
produces spin entropy by a partial trace over the momentum.

In this paper, we study the spin state of a particle moving in a gravi-
tational field to show its decoherence by the effects of general relativity [3].
Our result implies that even if the spin state is pure at one spacetime point,
it may become mixed at another spacetime point. This spin decoherence
is derived from the curvature of spacetime caused by the gravitational field.
Such a spacetime curvature entails a local description of spin by local Lorentz

frame due to a breakdown of the global rotational symmetry. The motion of a particle is thus accompanied by the change of frame, which can increase the spin entropy analogous to the case of special relativity. As an example, we consider a particle in a circular orbit around a static spherically-symmetric object using the Schwarzschild spacetime.

2 Formulation

Consider a wave packet of a spin-1/2 particle with mass m in a gravitational field. The gravitational field is described by a curved spacetime with metric in general relativity. Nevertheless, despite the spacetime curvature, we can locally describe this wave packet as if it were in a flat spacetime, since a curved spacetime locally looks like flat. More precisely, for any spacetime point we can find a coordinate system in which the metric becomes the Minkowski one. The coordinate transformation from a general coordinate system $\{x^\mu\}$ to this local Lorentz frame $\{x^a\}$ can be carried out using a vierbein (or a tetrad) $e_a{}^\mu(x)$ and its inverse $e^a{}_\mu(x)$ defined by [4]

$$e_a{}^\mu(x)\, e_b{}^\nu(x)\, g_{\mu\nu}(x) = \eta_{ab},$$
$$e^a{}_\mu(x)\, e_b{}^\mu(x) = \delta^a{}_b, \tag{1}$$

where $g_{\mu\nu}(x)$ is the metric in the general coordinate system and $\eta_{ab} \equiv$ diag$(-1,1,1,1)$ is the Minkowski metric with $a,b = 0,1,2,3$. The vierbein then transforms a tensor in the general coordinate system into that in the local Lorentz frame, and vice versa. For example, momentum $p^\mu(x)$ in the general coordinate system can be transformed into that in the local Lorentz frame via the relation $p^a(x) = e^a{}_\mu(x)\, p^\mu(x)$.

Therefore, we describe the wave packet as in the case of special relativity [1] using a local Lorentz frame at the spacetime point x^μ where the centroid of the wave packet is located. Since a momentum eigenstate of the particle is labeled by four-momentum $p^a = (\sqrt{|\vec{p}|^2 + m^2c^2}, \vec{p})$ and by the z-component σ $(=\uparrow, \downarrow)$ of spin [5] as $|\,p^a, \sigma\rangle$, the wave packet can be expressed by a linear combination

$$|\,\psi\rangle = \sum_\sigma \int d^3\vec{p}\, N(p^a)\, C(p^a, \sigma)\,|\,p^a, \sigma\rangle, \tag{2}$$

where

$$d^3\vec{p}\, N(p^a) \equiv d^3\vec{p}\, \frac{mc}{\sqrt{|\vec{p}|^2 + m^2c^2}} \tag{3}$$

is a Lorentz-invariant volume element. From the normalization condition

$$\langle p'^a, \sigma' \,|\, p^a, \sigma \rangle = \frac{1}{N(p^a)} \, \delta^3(\vec{p'} - \vec{p}) \, \delta_{\sigma'\sigma}, \tag{4}$$

the coefficient $C(p^a, \sigma)$ must satisfy

$$\sum_\sigma \int d^3\vec{p} \, N(p^a) |C(p^a, \sigma)|^2 = 1. \tag{5}$$

To obtain the spin state of this wave packet, we take the trace of the density matrix $\rho = |\psi\rangle\langle\psi|$ over the momentum,

$$\rho_{\rm r}(\sigma'; \sigma) = \int d^3\vec{p} \, N(p^a) \, \langle p^a, \sigma' \,|\rho|\, p^a, \sigma \rangle. \tag{6}$$

The spin entropy is then given by the von Neumann entropy of this reduced density matrix:

$$S = -\text{Tr} \left[\rho_{\rm r}(\sigma'; \sigma) \log_2 \rho_{\rm r}(\sigma'; \sigma) \right]. \tag{7}$$

Moreover, the spin operator for the wave packet is defined by

$$\hat{\vec{S}} = \frac{1}{2} \sum_{\alpha, \beta} \vec{\sigma}_{\alpha\beta} \int d^3\vec{p} \, N(p^a) \,|\, p^a, \alpha \rangle\langle p^a, \beta \,|, \tag{8}$$

with $\vec{\sigma}$ being the Pauli matrices. Note that this spin is not Dirac spin but Wigner one [6]. As is well known, Dirac spin, which corresponds to the index of 4-component Dirac spinor, is not a conserved quantity in a relativistic regime and thus is not suitable degree of freedom for labelling one-particle states. In contrast, Wigner spin is a conserved quantity suitable for labelling one-particle states, because it is defined using the particle's rest frame.

3   Decoherence

Suppose that the centroid of the wave packet is moving with four-velocity $u^\mu(x)$ normalized as $u^\mu(x)u_\mu(x) = -c^2$; this motion is not necessarily geodesic in the presence of an external force. After an infinitesimal proper time $d\tau$, the centroid moves to a new point $x'^\mu = x^\mu + u^\mu(x)d\tau$ and then the wave packet is described by the local Lorentz frame at the new point. This change in

the local Lorentz frame is represented by a Lorentz transformation $\tilde{\Lambda}^a{}_b(x) = \delta^a{}_b + \chi^a{}_b(x)d\tau$, where

$$\chi^a{}_b(x) = u^\mu(x)\left[e_b{}^\nu(x)\nabla_\mu e^a{}_\nu(x)\right]. \tag{9}$$

In addition to this change, the acceleration by an external force is also interpreted as a Lorentz transformation. Thus, the motion of the wave packet is equivalent to a Lorentz transformation $\Lambda^a{}_b(x) = \delta^a{}_b + \lambda^a{}_b(x)d\tau$, where [7]

$$\begin{aligned}\lambda^a{}_b(x) &= \chi^a{}_b(x) \\ &- \frac{1}{mc^2}\left[a^a(x)\,q_b(x) - q^a(x)\,a_b(x)\right]\end{aligned} \tag{10}$$

using the momentum and acceleration of the centroid in the local Lorentz frame

$$q^a(x) = e^a{}_\mu(x)\left[mu^\mu(x)\right], \tag{11}$$
$$a^a(x) = e^a{}_\mu(x)\left[u^\nu(x)\nabla_\nu u^\mu(x)\right]. \tag{12}$$

Note that even if the wave packet moves as straight as possible along a geodesic curve, this Lorentz transformation may be nonzero in general relativity because of the first term.

Since spin entropy is not invariant under a Lorentz transformation [1], neither is it invariant during the motion of the wave packet. Note that a Lorentz transformation rotates the spin of a particle through an angle that depends on the particle's momentum; this rotation is known as Wigner rotation. The momentum eigenstate $|p^a, \sigma\rangle$ thus transforms under the Lorentz transformation (10) as [5]

$$\begin{aligned}U(\Lambda(x))\,&|p^a, \sigma\rangle \\ &= \sum_{\sigma'} D_{\sigma'\sigma}(W(x))\,|\Lambda(x)p^a, \sigma'\rangle,\end{aligned} \tag{13}$$

where $D_{\sigma'\sigma}(W(x))$ is the 2×2 unitary matrix that represents a Wigner rotation given by [7]

$$\begin{aligned}W^i{}_k(x) &= \delta^i{}_k + \lambda^i{}_k(x)\,d\tau \\ &+ \frac{\lambda^i{}_0(x)\,p_k - \lambda_{k0}(x)\,p^i}{p^0 + mc}\,d\tau\end{aligned} \tag{14}$$

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with $i, k = 1, 2, 3$. Taking the trace of the density matrix

$$\rho' = U(\Lambda(x))|\psi\rangle\langle\psi|U(\Lambda(x))^\dagger$$

over the momentum, we obtain the spin state $\rho'_{\mathrm{r}}(\sigma'; \sigma)$ and the spin entropy S' of the wave packet in the local Lorentz frame at the new point x'^μ. However, the spin has been entangled with the momentum by the Lorentz transformation (10), since the Wigner rotation (14) of spin depends on the momentum. Due to this entanglement, the new entropy S' is not, in general, equal to the original one S. This implies that the spin state may decohere during the motion of the wave packet by the effects of general relativity.

4   Example

As an example in general relativity, we consider the Schwarzschild spacetime, which is the unique static spherically-symmetric solution of Einstein's equation in vacuum. In the spherical coordinate system (t, r, θ, ϕ), the metric is given by

$$g_{\mu\nu}(x)dx^\mu dx^\nu = -f(r)c^2 dt^2 + \frac{1}{f(r)}dr^2$$
$$+ r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{15}$$

where $f(r) = 1 - (r_s/r)$ with the Schwarzschild radius r_s. In the Schwarzschild spacetime, we introduce an observer at each point who is static with respect to the time t using a static local Lorentz frame. The vierbein (1) is then

$$e_0{}^t(x) = \frac{1}{c\sqrt{f(r)}}, \quad e_1{}^r(x) = \sqrt{f(r)},$$
$$e_2{}^\theta(x) = \frac{1}{r}, \quad e_3{}^\phi(x) = \frac{1}{r\sin\theta}. \tag{16}$$

Suppose that the centroid of the wave packet is moving along a circular trajectory of radius r ($> r_s$) with a constant velocity $rd\phi/dt \equiv v\sqrt{f(r)}$ on the equatorial plane $\theta = \pi/2$ (see Fig. 1). The four-velocity of the centroid is then given by

$$u^t(x) = \frac{\cosh\xi}{\sqrt{f(r)}}, \quad u^\phi(x) = \frac{c\sinh\xi}{r}, \tag{17}$$
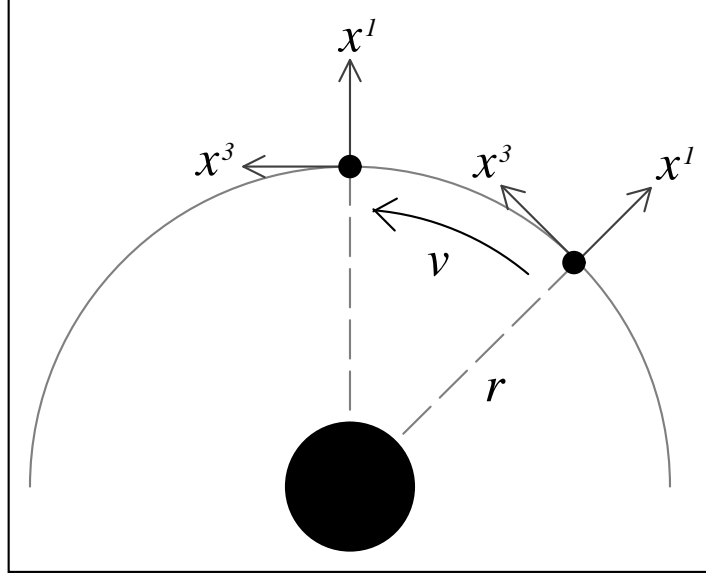
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Figure 1: A wave packet (small circle) going around a static spherically-symmetric object (large circle). The 1- and 3-axes of the local Lorentz frame are illustrated at the initial and final points.

where ξ is defined by $\tanh \xi = v/c$. We assume that at the initial point the wave packet has the definite values $\sigma = \uparrow$ and $p^1 = p^2 = 0$ but is Gaussian in p^3 with width w:

$$
|C(p^a, \sigma)|^2 = \frac{1}{N(p^a)} \delta(p^1) \, \delta(p^2) \, \delta_{\sigma, \uparrow}
$$
$$
\times \frac{1}{\sqrt{\pi} w} \exp \left[-\frac{(p^3 - q^3(x))^2}{w^2} \right], \tag{18}
$$

where $q^3(x) = mc \sinh \xi$ is the momentum of the centroid (11) along the 3-direction.

Clearly, the spin entropy of this wave packet is zero at the initial point, since the spin is separable from the momentum. However, after a proper time τ of the particle, spin entropy is generated by the gravity and acceleration, i.e., by the first and second terms in Eq. (10). Figure 2 shows the generated spin entropy S as a function of the proper time τ in the case of $v/c = 0.8$,
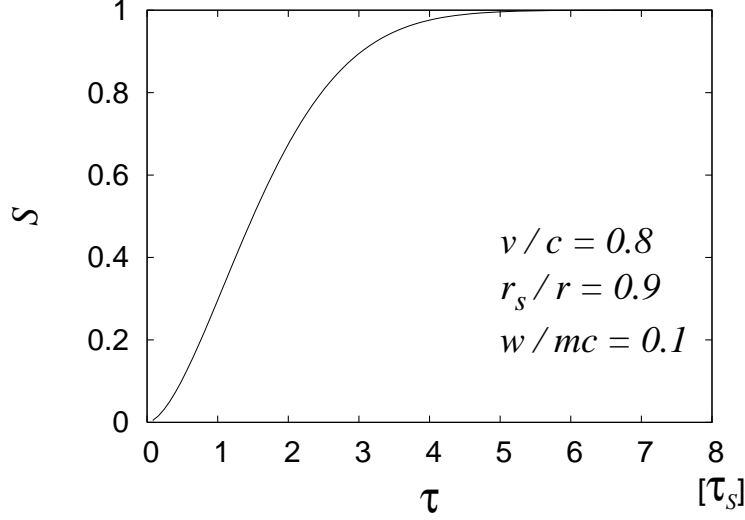
Figure 2: The spin entropy S at $v/c = 0.8$, $r/r_s = 0.9$, and $w/mc = 0.1$ as a function of the proper time τ normalized by $\tau_s \equiv mr_s/w$.

$r/r_s = 0.9$, and $w/mc = 0.1$. The spin state of the wave packet decoheres to a mixed state and becomes maximally mixed ($S \to 1$) in the limit of $\tau = \infty$. The characteristic decoherence time is given by the inverse of

$$\tau_d^{-1} \equiv \frac{w(\cosh\xi - 1)}{mr}\left|1 - \frac{r_s}{2rf(r)}\right|\sqrt{f(r)}. \tag{19}$$

Figure 3 shows this value τ_d^{-1} as a function of r_s/r when $v/c = 0.8$. No decoherence occurs ($\tau_d \to \infty$) at the spatial infinity $r \to \infty$, whereas extremely rapid decoherence occurs ($\tau_d \to 0$) near the Schwarzschild radius $r \to r_s$. The spin state does not decohere also at $r = 3r_s/2$, because the first term in Eq. (10) is canceled by the second term.

Of course, this spin decoherence is very slow in the gravitational field of the earth $r_s \sim 1\,\mathrm{cm}$. For example, when a wave packet is at rest in the International Space Station going around the earth, $v \sim 7.7\,\mathrm{km/s}$ and $r \sim 6800\,\mathrm{km}$, the characteristic decoherence time is $\tau_d \sim 2.2 \times mc/w$ years.
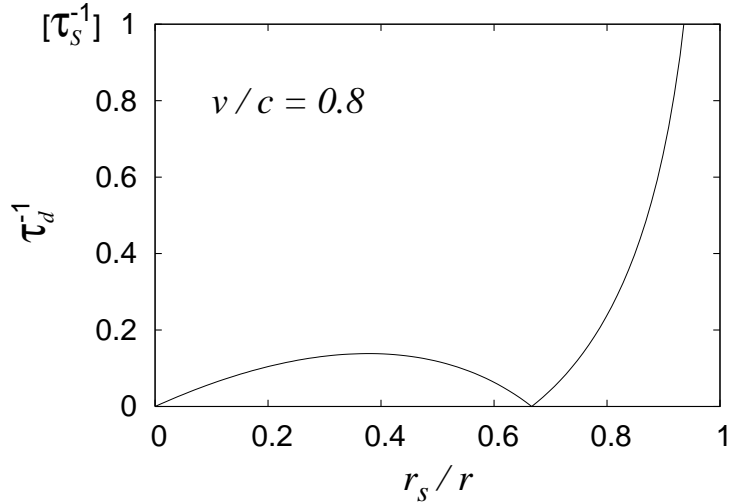
Figure 3: The inverse of the characteristic decoherence time τ_d^{-1}, normalized by τ_s^{-1}, as a function of r_s/r at $v/c = 0.8$.

5   Conclusion

We have shown that spin entropy is generated when a particle moves in a gravitational field. The spin state evolves into a mixed state even if the particle moves as straight as possible along a geodesic curve. This decoherence is due to the spacetime curvature by gravity.

Acknowledgments

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